Solid Mechanics

Stress

What you'll learn:

- What is stress?
- Why stress is important?
- What are normal and shear stresses?
- What is strain?
- Hooke's law (relationship between stress and strain)
- Stress – strain diagram
- Stress concentration

Motivation

- High heel shoes:

Have you ever tried high heel shoes? If no, it worth to try it at least once, to understand one of the major concepts in engineering. Do you know what would happen if you walk on a soft ground wearing high heels? Can you explain the differences between 2 above shows?

- Pet Collar
Do you have a pet? Do you use collar? Have you ever thought which kind of collar is better? What would happen if it is so thin? Will you use a thin rope instead of a collar?

- Knife

Have you ever cut your hand by a paper? Have you ever tried to cut something by a knife but it wasn't sharp enough? What are the similarities between the knife and a piece of paper?

- Weight lifting

You may watch Olympic weight lifting competition. Have you ever thought that, they are in static equilibrium so why they seem under a lot of pressure?

- Book lifting

Try to hold a heavy book by one hand, could you hold it? Congratulations. Now, first of all have a knife on your hand and put the book on top of the knife. What happened? Do you dare to do it? Why?
**Definition**

Stress is defined as force divided by cross sectional area. There are 2 different types of stress, shear stress and normal stress. The *normal stress* \( \sigma \) (is denoted by the Greek letter sigma) acts perpendicular to the selected plane.

\[
\sigma = \frac{P}{A} \quad \text{Equation 1}
\]

Figure 1 shows the forces on a rod. This rod is in a static equilibrium. An axial force \( F \) is applied to each end of this rod. The cross section of this rod is \( A \). The force \( P \) acts perpendicular to the cross-sectional area. Normal stress in this rod is defined by equation 1.

![Figure 1 Normal stress](image1.png)

Figure 2 shows the same rod with the forces applied parallel to the rod’s cross sectional area. The stress here is defined as shear stress. However, it can be computed by equation 1.

![Figure 2 Shear stress](image2.png)
Stress Units

In SI metric units, with \( P \) expressed in Newtons (N) and \( A \) in square meters (m\(^2\)), the stress \( \sigma \) will be expressed in N/m\(^2\). This unit is called *Pascal* (Pa). However, one finds that the pascal is an exceedingly small quantity and that, in practice, multiples of this unit must be used, namely, the kilopascal (kPa), the megapascal (MPa) and the gigapascal (GPa). We have:

\[
\begin{align*}
1 \text{ kPa} &= 10^3 \text{ Pa} \\
1 \text{ MPa} &= 10^6 \text{ Pa} \\
1 \text{ GPa} &= 10^9 \text{ Pa}
\end{align*}
\]

When U.S. customary units are used, the force \( P \) is usually express in pounds (lb) or kilopounds (kip), and the cross sectional area \( A \) is square inches (in\(^2\)). The stress \( \sigma \) will then be expressed in ponds per square inch (psi) or kilopounds per square inch (ksi).

The cross section of the bar in Figure 1 is circular, but the the equation 1.1 is valid for any cross-sectional area of a member of any shape (i.e., circular, rectangular, triangular, etc.).

Tensile and Compressive Stress

When the force vectors are directed away from each other (Figure 1), the bar is stretched. The normal stress associated with this force configuration is referred to as *tensile stress* because the forces place the body in tension. Conversely, if the force vectors are directed toward each other, the bar is compressed. The normal stress associated with this force configuration is referred to as *compressive stress* because the forces place the body in compression. These two force configurations are illustrated in Figure 3.
A positive sign for stress will be used to indicate a tensile stress (member in tension) and a negative sign to indicate a compressive stress (member in compression). Since some of the material can withstand just one type of stress (tensile or compressive), the type of normal stress is of prime importance. For example concrete is stronger in compression than in tension. There it is mostly used in applications where the stresses are compressive. Like columns in bridges and buildings.

Example:

A column of square cross section is subjected to a compressive load of 6 kN. If the width of the column is 25 cm, find the stress on the column.

Solution:

\[ A = 25 \text{ cm} \times 25 \text{ cm} = 625 \text{ cm}^2 = 625 \times 10^{-4} \text{ m}^2 = 6.25 \times 10^{-6} \text{ m}^2 \]

\[ \sigma = \frac{F}{A} = \frac{6 \times (10^3 \text{ N})}{(6.25 \times 10^{-6} \text{ m}^2)} = 960 \text{ MPa} \]

Example:

A rod with the diameter of 25 cm is subjected to 500N. Find the stress on the rod.

Solution:

\[ A = \pi \frac{d^2}{4} = (3.14) \times \frac{25^2 \times 10^{-4}}{4} = 490.625 \text{ m}^2 \]

\[ \sigma = \frac{F}{A} = \frac{500 \text{ N}}{(490.625 \times 10^{-4} \text{ m}^2)} = 1.2 \times 10^4 \text{ Pa} \]

The shear stress \( \tau \) (is denoted by the Greek letter tau) acts parallel to the selected plane. The mathematical presentation of shear stress is the same as normal stress. So, we can use equation 2 here too.

\[ \tau = \frac{P}{A} \quad \text{Equation 2} \]

How to recognize shear stress than normal stress?

The answer is: find your force direction and identify your structure's area. What's the angle between force and area? Is this zero, 90 or something in between? (Figure 4)
When θ = 0 there is only shear stress, for θ = 90 we just have normal stress and for θ = 45 we have both shear and normal stresses.

**Example:**
We need to hang something from ceiling, find stress applied to ceiling? Which kind of stress is that? Do the same problem when you need to hang on your subject from wall.
W=50 N, Ceiling Area= 4 m², wall Area= 5 m²

**Solution:**
First we need to identify the force and its direction. Here weight is the force and we know its direction is always toward ground. In the first part of the question we will hang on our subject from ceiling (Figure 5 a). Now you realize that the force (weight) is perpendicular to the ceiling, consequently we will have normal stress.

![Figure 4. Force angle with the plane](image)

![Figure 5. Hanging a subject from ceiling or wall](image)
\[ \sigma_N = \frac{F}{A} = \frac{50 \text{ (N)}}{4 \text{ (m}^2\text{)}} = 12.5 \text{ Pa} \]

This stress is a normal stress and it is tensile.

For the second part (Hanging from wall), since the force doesn’t change we have the same force as the part one. But, now our area is changed and as you can see in Figure (5 b) the force is parallel to the area. It means now we have shear stress, which we can find it with the same equation:

\[ \tau = \frac{F}{A} = \frac{50 \text{ (N)}}{5 \text{ (m}^2\text{)}} = 10 \text{ pa} \]

In conclusion, you see that, hanging something from a wall creates a shear stress on the wall, since the weight of the object is acting parallel to the wall, as opposed to hanging something from the ceiling which creates a normal stress on the ceiling, since the weight is acting perpendicular to the ceiling.
Strain

Motivation

What’s the consequence of wearing high heels instead of flat shoes and walking in soft floor?
What would happen if you try to cut a metal using your butter knife?
Did you ever stretch a rubber band and notice that its length changes?

Definition

No material is perfectly rigid. When they subjected to external force they change their size and shape. This change in the shape or size of the body is defined as deformation. All materials—steel, concrete, wood, and other structural materials—deform to some extent under applied forces, but the deformations are usually too small to detect visually.
Figure 6 shows the deformation of a rod. Deformation usually denoted by the Greek letter delta (δ).

Prior to applying force the bar has a length L1. The applied tensile force causes the bar to increase its length to L2. The difference between two lengths (prior and after applying load) is deformation (δ). Since the change in length is caused by normal stress the quantity δ is called the normal deformation or axial deformation.
In order to normalize the change in size or shape of body with respect to the body’s original geometry, engineers use a quantity called strain. There are two types of strain, normal strain and shear strain.
Normal strain $\varepsilon$ (Greek letter epsilon) is defined as the *normal deformation divided by the original length* $L$:

$$\varepsilon = \frac{\delta}{L} \quad \text{Equation 3}$$

**Strain units**

Because strain is a ratio of two lengths, it is a dimensionless quantity. It is customary, however, to express strain as a ratio of two length units. In the SI unit system, strain is usually expressed in units of μm/m because, as mentioned before, deformations are typically small. In the English unit system, strain is usually expressed in units of in/in. Since strain is a dimensionless quantity; it is sometimes expressed as a percentage. The normal strain illustrated in Figure 3 is for a body in tension, but the definition given by Equation (3) also applies to bodies in compression.

**Example:**

In example 1 find the strain when the column has the 1.2 m height before applying force and 1.199 m after applying force.

**Solution:**

$$\delta = \text{deformation} = L_2 - L_1 = 1.199 - 1.2 = -0.001 \text{ m}$$

$$\varepsilon = \frac{\delta}{L} = -0.001/1.2 = -8.3 \times 10^{-4}$$

Strain has the negative sign since the stress is compressive and the height of the column reduced.

**Strain caused by shear stress:**

Shear stress can cause deformation. Figure 7 shows the shear stress and its deformation on a plane. This plane is subjected to the shear stress $\tau$. It observed to deform into a parallelogram. The deformation on this plane is $\gamma$ (Greek letter Gamma).

![Figure 7. Shear strain](image-url)
Example:

Consider a rectangular block as shown in Figure 8. The lower plate is fixed, while the upper plate is subjected to a horizontal force $F$. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine the shearing strain in material.

Solution:

Because of the shear stress point $B$ moved to point $C$. According to the definition the shearing strain is $\gamma$, which is equal to the angle formed by vertical line (AB) and line (AC). Noting that this is very small angle and recalling that it should be expressed in radians, we write:

$$\gamma \approx \tan \gamma = \frac{0.04 \text{ in}}{4 \text{ in}} = 0.01 \text{ rad}$$
Hooke’s Law

In 1678 the English mathematician Robert Hooke (1635—1703) stated a law which is known as Hooke’s law of elasticity. He discovered that the extension of a string is directly proportional to the load applied to the spring. This law is expressed mathematically as:

\[ F = kx \]  

Equation 4

Where \( F \) is force, \( x \) is displacement, and \( k \) is a constant of proportionality called the spring constant. Equation 4 is valid only if the spring can return to its original length after the force is removed.

There is another form of Hooke’s law for engineering materials which has the same mathematical form of equation 4 but is expressed in terms of stress and strain (Equation 5).

\[ \sigma = E\varepsilon \]  

Equation 5

This form of the Hook’s law, Equation (5), states that the stress \( \sigma \) in a material is proportional to the strain \( \varepsilon \). Here there is the same limitation which says, this law is true only for the material which returns to its original size after the removal of the force. These materials that obey Hook’s law are known as linear-elastic or "Hookean" materials.

You already know that what are \( \sigma \) and \( \varepsilon \) in equation 4. The constant of proportionality \( E \), in this equation is called the modulus of elasticity or Young’s modulus, after the English mathematician Thomas Young (1773—1829). Since strain is a dimensionless quantity the modulus of elasticity or Young’s modulus has the same units as stress.

As we will see later in this lesson when you do a set of experiment on a linear elastic material and you measure the strain for specific applied stress you can plot stress strain data points on a linear scale. In this diagram based on Hook’s Law we expect to get a straight line. \( E \) would be the slope of the best straight line fitted through the points.

The equation 4 can be express in terms of force and deformation. Take a piece of paper and try to find that. You should have end up to this equation:

\[ \Delta L = \frac{F L}{A E} \]
Example

A 200 kg engine block hangs from a cable (Figure 9). Find the normal stress and the axial deformation in cable. The cable is 0.7 m long and have a diameter of 4 mm. the cable is steel with a modulus of elasticity of $E = 200 \text{ GPa}$.

![Figure 9. Engine block hanging from cable][1]

Solution:

$F = \text{engine weight} = m \times g = 200 \text{ (kg)} \times 9.81 \text{ (m/s}^2) = 1962 \text{ N} = 1.962 \text{ kN}$

$A = \pi r^2 = 3.14 \times (2 \text{ mm})^2 = 12.56 \text{ mm}^2 = 12.56 \times 10^{-6} \text{ m}^2$

$\sigma = \frac{F}{A} = \frac{1962 \text{ (N)}}{(12.56 \times 10^{-6} \text{ m}^2)} = 156.2 \times 10^6 \text{ Pa} = 156.2 \text{ MPa}$

$\sigma = E \varepsilon$

$\varepsilon = \frac{\sigma}{E} = \frac{156.2 \text{ (MPa)}}{200 \text{ (GPa)}} = 0.781 \times 10^{-3}$

$\varepsilon = \frac{\delta}{L}$

$\delta = \varepsilon \times L = 0.781 \times 10^{-3} \times 0.7 \text{ (m)} = 0.546 \times 10^{-3} \text{ m} = 0.546 \text{ mm}$

[1]: https://example.com
Earlier we mentioned about the stress-strain diagram. This is a graph of stress as a function of strain. This graph is derived from measuring the load applied on the sample, and strain, derived from measuring the deformation of the sample, i.e. elongation, compression, or distortion. This test is called tensile test. Figure 10 shows a typical stress strain diagram. The nature of the curve varies from material to material and even on the same material depends on the temperature of the specimen and the speed of the loading may yield different results. However it is possible to find some common characteristics among the Stress-Strain Diagram of various groups of materials. The very useful fact about this curve is that, it does not depend upon the dimension of the particular specimen used. “The upper stress limit of the linear relationship described by Hooke’s law is called the proportional limit, labeled point $A$. At any stress between point $A$ and the elastic limit, labeled point $B$, stress is not proportional to strain, but the material will still return to its original size after the force is removed. For many materials, the proportional and elastic limits are very close together. Point $C$ is called the yield stress or yield strength. Any stress above the yield stress
will result in plastic deformation of the material (i.e., the material will not return to its original size, but will deform permanently). As the stress increases beyond the yield stress, the material experiences a large increase in strain for a small increase in stress.) At about point D, called the ultimate stress or ultimate strength, the cross-sectional area the material begins to decrease rapidly until the material experiences fracture at point E.” [1]

**Example**

A rod of aluminum 6061-T6 has a square cress section measuring 6mm by 6mm. Using the yield stress as the failure stress, find the maximum tensile load that the rod can sustain. The yield stress of aluminum 6061-T6 is 240 MPa.

**Solution:**

The cross section is square. Area = 6 (mm) * 6 (mm) = 36 mm² = 36* 10⁻⁶ m²

\(1 \text{ mm} = 10^{-3} \text{ m}, 1\text{ mm}² = 10^{-6} \text{ m}²\)

\(\sigma = \frac{F}{A} \rightarrow F = \sigma * A\)

If the maximum tensile load applied to the rod it will reach the yield point. So, to find the maximum tensile load:

\(F = \sigma * A = 240 \text{ (MPa)} * 36 \text{ (mm}²\text{)} = 240 \times 10^6 \text{N/m}² \times 36 \times 10^{-6} \text{ m}² = 8640 \text{ N} = 8.64 \text{ kN}\)
Stress concentration

Airplane window (de Havilland Comet)


“Up to now we assumed that the members’ cross sections were uniform throughout. However, most real machine parts will have varying cross sections. For example, shafts often are stepped to different diagrams to accommodate bearing, gears, pulleys, etc. A shaft may have holes for the attachment of other parts. Any of these changes in cross-sectional geometry will cause localized stress concentration.

Figure 11 shows that stress concentration introduced by notches and fillets in a flat bar subjected to a bending moment. The stress effects were measured and the stress distribution is shown in the part when loaded. Note that, at the right end of the part where the cross section is uniform, the fringe lines are straight, of uniform width, and equispaced. This indicates a linear stress distribution across this portion of the bar. But, at the fillet where the width of the part is reduced from D to d, the fringe lines indicate a disruption and concentration of stress at this sudden change in geometry. The same effect is seen near the left-hand end around the two notches. The nonlinear stress distribution across the
section at the notches is plotted on the lower figure. This is experimental evidence of the existence of stress concentration at any change in geometry. Such geometry changes in a part are often called “stress-raisers” and should be avoided or at least minimized as much as possible in design. Unfortunately, it is not practical to eliminate all such stress-raisers, since such geometric details are needed to connect mating parts and provide functional part shapes.

The amount of stress concentration in any particular geometry is denoted by a geometric stress-concentration factor $K_t$ for normal stress. The maximum stress at local stress-raisers is then defined as:

$$\sigma_{\text{max}} = K_t \sigma_{\text{nom}}$$

Where $\sigma_{\text{nom}}$ is the nominal stress calculated for the particular applied loading and net cross section, assuming a stress distribution across the section that would obtain for a uniform geometry. Note that the nominal stresses are calculated using the net cross section which is reduced by the notch geometry. “ [2]

**Example**

A flat bar with a transverse hole is subjected to a 500 kN force in axial direction, as shown in figure 12. In this figure $w = 20$ cm, $d = 2$ cm, $h = 4$ cm. Find the maximum stress in the bar ($K_t = 2.5$)

![Figure 12. Bar with a hole [2]](image)

**Solution:**

The cross section with the hole has the less area and because of geometry change there is stress concentration in this section.

$$A = w * h - d * h = (w-d)*h = (20 - 2) * 4 = 72 \text{ cm}^2$$

$$\sigma = F/A = 500 * 10^3 \text{ N/72 * 10}^{-4} \text{ m}^2 = 69.4 \text{ MPa}$$

$$\sigma_{\text{max}} = \sigma * K_t = 69.4 \text{ MPa} * 2.5 = 173.5 \text{ MPa}$$
References