GENERALLY APPLICABLE SOLUTIONS OF ZOEPPRITZ' AMPLITUDE EQUATIONS

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ABSTRACT

A crustal model, in addition to satisfying travel-time data, should agree with the observed distribution of amplitude in time and range. The study of observed amplitudes in seismic crustal work depends directly on the partition of the amplitude of an incident wave front at an interface. The Zoeppritz equations describe this partition for a plane wave and while they are simple simultaneous linear equations, their solution by hand methods is tedious and receptive to errors. Solutions of these equations have been obtained by numerical methods for both the incident $P$ and $SV$ cases and are presented in the form of curves showing the ratio of the amplitude of the resultant ray to that of the incident ray plotted against the angle of incidence. By solving the equations for velocity ratios ranging from 0.25 to 4.0, families of curves covering most realistic situations were obtained.

That generally applicable solutions of these equations can be presented in curve families ranging over velocity contrast alone, is demonstrable from the studies of the velocity vs. density relation made by Nafe and Drake and independently by Woollard.

The curves presented permit rapid, quantitative solution of the amplitude partition at an interface, thereby, greatly facilitating the study of wave train modifications.

INTRODUCTION

Curiosity about the importance of $P$ and $S$ conversions in crustal seismic measurements led us to an examination of the Zoeppritz amplitude relations. Initially, we wondered whether conversions of a wave front approaching the surface through a multiply-layered crust could explain later portions of a seismogram and help support or challenge the concept of layering as opposed to other theories of crustal inhomogeneity. We found that the Zoeppritz solutions provided much insight into our particular problem and that it might be instructive to examine them in greater detail.

As a result, the Zoeppritz equations were machine solved and machine plotted for a wide range of parameters and have since suggested several interesting approaches to the investigation of layered models. The purpose of this article is to present these solutions in graphical form.

There are plentiful references to the Zoeppritz equations in the literature (Zoeppritz, 1919; Macelwane, 1933 and 1936; Richter, 1958; Jakosky, 1950; Steinhart and Meyer, 1961; and Nafe, 1957). A thorough discussion of the matter at hand is contained in Steinhart and Meyer (1961, pp. 90–113), so, instead of reiterating their remarks, the following is intended to be an amplified caption for the enclosed families of curves.

BACKGROUND

The equations were taken after Richter (1958). With the notation used in the graphs, they are:

for incident $P$,

$$(A - C) \sin a + D \cos b - E \sin e + F \cos f = 0$$
\[(A + C) \cos a + D \sin b - E \cos e - F \sin f = 0\]

\[-(A + C) \sin 2a + D \frac{V_1}{U_1} \cos 2b + EK \left( \frac{U_2}{U_1} \right)^2 \frac{V_1}{V_2} \sin 2e - FK \left( \frac{U_2}{U_1} \right)^2 \frac{V_1}{V_2} \cos 2f = 0\]

\[-(A - C) \cos 2b + D \frac{U_1}{V_1} \sin 2b + EK \frac{V_2}{V_1} \cos 2f + FK \frac{U_2}{V_1} \sin 2f = 0\]

and for incident SV,

\[(B + D) \sin b + C \cos a - E \cos e - F \sin f = 0\]

\[(B - D) \cos b + C \sin a + E \sin e - F \cos f = 0\]

\[(B + D) \cos 2b - C \frac{U_1}{V_1} \sin 2a + EK \frac{U_2^2}{U_1 V_2} \sin 2e - FK \frac{U_2}{U_1} \cos 2f = 0\]

\[-(B - D) \sin 2b + C \frac{V_1}{U_1} \cos 2b + EK \frac{V_2}{U_1} \cos 2f + FK \frac{U_2}{U_1} \sin 2f = 0.\]

Where \(A\) and \(B\) are the amplitudes of P and SV, incident at angles \(a\) and \(b\) respectively and \(C, D, E, F,\) are the amplitudes of the resultant rays, reflected P, reflected SV, refracted P and refracted SV, at angles \(a, b, e,\) and \(f.\) \(V_1, U_1, V_2\) and \(U_2\) are the P and SV velocities in the first and second layers respectively. \(K\) is the ratio of the density of the second layer to the density of the first. The angles \(a, b, e, f\) are related by Snell’s Law:

\[
\frac{\sin a}{V_1} = \frac{\sin b}{U_1} = \frac{\sin e}{V_2} = \frac{\sin f}{U_2}.
\]

Clearly, with incident SV, whenever \(\sin b > \left(\frac{U_1}{V_1}\right),\) \(\sin a > 1\) and \(a\) must be an imaginary angle. Similarly, \(e\) and \(f\) can also become imaginary with incident SV. With incident P, \(e\) and \(f\) alone may be imaginary.

When any angle, \(m,\) becomes imaginary,

\[\cos m = \pm i (\sin^2 m - 1)^{1/2}\]

and each equation breaks into a real and imaginary part, yielding a system of rank eight. The sign of the imaginary cosine is taken to conform with the derivation in Jeffreys (1952, p. 31), and the equations are solved for both the real and imaginary parts of the amplitudes. The square root of the sum of the squares of the real and imaginary parts gives the actual, physical amplitude.

When a particular ray has an imaginary angle, its amplitude is in general non-zero. This amplitude corresponds to a boundary effect which decreases exponentially away from the interface and which moves along the interface with the velocity of the intersection of the wave front and the interface. A “pseudo-surface wave” of this nature could not be said to propagate since it is essentially a forced vibration.
caused by the impinging wave front on the interface and its velocity is wholly dependent on the angle of incidence. Hence, for normal application no significance should be placed on those portions of the curves which represent rays having imaginary angles. In this presentation these portions are indicated by dotted lines. In addition to this convention, there are also a few lines which are dashed for their
entire real range; these dashed lines are intended as "tracers" to aid following the trend of the curves in areas of visual confusion.

**Generality of Results**

In order to keep within a realistic range of velocities, one velocity was held constant at 8.0 km/sec and the other was varied from 2.0 to 8.0 km/sec. With $V_2$ held at 8.0, $V_1$ was varied to give 3.5 degree increments in the critical angles. For the case where $V_1 > V_2$, the same velocities were retained and the roles of $V_1$ and $V_2$ were simply reversed. The specific values considered and corresponding curve numbers are summarized in table 1. The density ratios corresponding to these velocity ratios were taken from the curve showing the observed ratio of compressional velocity and density by Nafe and Drake (Talwani et al., 1959). This curve is very nearly linear for velocities greater than 5 km/sec, with only a slight departure from linearity below that value (see figure 2). The effect of this local nonlinearity on the general validity of the Zoeppritz solutions was investigated by solving the equations twice, holding the velocity ratio constant and selecting the velocities to give a great disparity in density ratios. The resultant curves were superimposed and examined for differences. Figures 3 and 4 show the cases of greatest variation due to this disparity. Note that the fundamental forms of the curves are unchanged, while the differences in absolute value although noticeable, are small. Thus, the assumption of the Nafe and Drake velocity/density relation permits the general application of Zoeppritz solution curves ranging over velocity ratio alone.

A similar method was used to investigate briefly the effect of variations in Poisson's ratio (see figure 5), which appears to be small enough to neglect completely. In this presentation, Poisson's ratio is taken to be 0.25.

**Numerical Technique**

The program was written in FORTRAN for a CDC 1604 computer. The equations were solved at one degree intervals using a modified matrix inversion technique. One half degree solutions in a five degree neighborhood of the critical angle provided additional detail.
OBSERVED RELATION OF COMPRESSIONAL VELOCITY

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**Fig. 2**

- **WOOLLARD (1959)**
- **NAFE & DRAKE (TALWANI ET AL., 1959)**

Arrows indicate velocities used in ratio test.
EFFECT OF GROSS NON-LINEARITY OF VELOCITY-DENSITY RELATION WITH INCIDENT P

\[ \frac{V}{V_d} = 0.600 \]

\[ \frac{\rho}{\rho_d} = 0.878 \]

\[ \frac{\rho}{\rho_d} = 0.764 \]

FIG. 3
EXTREME EXAMPLES # EFFECT OF NON-LINEARITY
OF VELOCITY DENSITY RELATION UNDER INCIDENT SV

\[ \frac{V}{V_2} = 0.600 \]

\[ \theta = 60^\circ \]

\[ \theta = 75^\circ \]

\[ \theta = 90^\circ \]

**Fig. 4**
EFFECT OF CHANGES IN POISSON'S RATIO ON AMPLITUDE COEFFICIENTS

\[
\frac{v_1}{v_2} = 0.600
\]

\[
\sigma = 0.25
\]

\[
\sigma = 0.27
\]

INCIDENT P

REFLECTION P

FIG. 5
SOLUTIONS OF ZOEPPRITZ' EQUATIONS

Fig. 6

INCIDENT — P
REFLECTED — P
LINES 1-6

$V_1 < V_2$
INCIDENT-P
REFLECTED-P
LINES 7-21

\( v_1 < v_2 \)

Fig. 7
INCIDENT P
REFLECTED S

$V_{1} < V_{2}$

Fig. 8
INCIDENT P
REFRACTED P

$V_1 < V_2$

Fig. 9
Fig. 10

INCIDENT- P
REFRACTED- S
CURVES 1-8

$V_1 < V_2$

ANGLE OF INCIDENCE (DEGREES)
FIG. 11

INCIDENT-S
REFLECTED-P
CURVES 1-8
$v_1 < v_2$

ANGLE OF INCIDENCE (DEGREES)
INCIDENT-S
REFLECTED-P
CURVES 6-16

\[ v_1 < v_2 \]

\( C / B \)

ANGLE OF INCIDENCE (DEGREES)

FIG. 12
INCIDENT S
REFLECTED S
CURVES 1-9

V₁ < V₂

Fig. 13
FIG. 15

INCIDENT S
REFRACTED P
CURVES 1-11

$V_1 < V_2$

$\frac{E}{B}$

ANGLE OF INCIDENCE (DEGREES)
Fig. 16
\text{FIG. 17}
Figure 18
INCIDENT-P
REFLECTED-P
CURVES 33-45

$V_1 > V_2$

Fig. 19
SOLUTIONS OF ZOEPPRITZ' EQUATIONS

Fig. 20
INCIDENT - P
REFRACTED - P
CURVES 24-44
\( V_1 > V_2 \)

Fig. 21
INCIDENT - P
REFRACTED - S
CURVES 24 - 44
\nu_i > \nu_2

Fig. 22
INCIDENT S P REFLECTED CURVES 24-33

\( v_1 > v_2 \)

Fig. 23
INCIDENT S
REFLECTED P
CURVES 34-45

V₁ > V₂

FIG. 24
INCIDENT-S
REFLECTED-S
CURVES 24-38
$V_1 > V_2$

FIG. 25
FIG. 26
INCIDENT-S  
REFRACTED-P  
CURVES 31-45  
$V_1 > V_2$
SOLUTIONS OF ZOEPPRITZ' EQUATIONS

Fig. 28

INCIDENT-S
REFRACTED-S
CURVES 24-31

$V_1 > V_2$

$F / B$

ANGLE OF INCIDENCE (DEGREES)
INCIDENT-S
REFRACTED-S
CURVES 31-45

\[ v_1 > v_2 \]

F / B

NOTE: UNITS 20-44 OMITTED FOR CLARITY - FOLLOW UNIFORMLY

ANGLE OF INCIDENCE (DEGREES)

FIG. 29
The program output was both printed and punched in a format acceptable to an X-Y plotter. Over 34,300 point solutions were required to produce the solutions.

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